In the following discussion, we follow Euclid and assume that the measure of an angle is positive and less than two right angles.

Definition.
An *arc* is a connected subset of a circle. An arc has endpoints on the circle. The *base* of the arc is the chord between the endpoints.

A *circle segment* given by an arc of a chord is the region in the plane bounded by an arc and its base. Given a chord, an arc, or a segment, the other two are determined.

An arc is a *major arc* if the center of the circle is inside the corresponding segment, which is called the *major segment*.

An arc is a *minor arc* if the center of the circle is outside the corresponding segment, which is called the *minor segment*.

An arc is a *semicircle* if the base is a diameter.the arc from point A to point B.

Two arcs are *dual* if they are distinct arcs on the same circle with the same endpoints.

If A and B are points on the circle, the minor arc from A to B is denoted $\widehat{AB}$. The semicircle or major arc is endpoints A and B is denoted $\widehat{ACB}$, where C is a point in the arc between A and B.

A *central angle* of a circle is an angle whose vertex is the center of the circle.

An *inscribed angle* of a circle is an angle inside the circle whose vertex is on the circle.

The *base* of a central or inscribed angle is the chord connecting the points on the circle where the sides of the angle, if extended, intersect the circle. The *base arc* of a central or inscribed angle is the the arc contained by the angle.

The *measure* of a minor arc is the measure of the central angle which corresponds to its base. The measure of a major arc is $360°$ minus the measure of its dual. The measure of a semicircle is $180°$.

We now summarize parts of Euclid Book III, Propositions 20, 23, 24, 25, 26, 27, 28, 29, and 31.

Euclid proves the Central Angle Theorem only in the case of inscribed angles in the major segment, but we state it in complete generality.

Proposition 20. Central Angle Theorem
The measure of an angle inscribed in a circle is half the measure of its base arc.

The angle is acute if and only if it lies in the major segment of its base.

The angle is obtuse if and only if it lies in the minor segment of its base.

The angle is right if and only if its base is a diameter.

Proof. Euclid proves the Central Angle Theorem in the case that the angle lies in the major segment. Thales' Theorem supplies the case that the angle lies on a semicircle. We provide the case that the angle lies in the minor segment.

Let $\widehat{AC}$ be a minor arc on $\odot O$, and let B be a point on the arc and let Z be a point on its dual. We claim that $m\angle ABC = \frac{1}{2} m\widehat{AZC}$.

Let $\alpha = m\angle AOB$ and $\beta = m\angle OCB$. We know that $OA = OB = OC$, since they are radii of $\odot O$. Thus, $\triangle AOB$ and $\triangle BOC$ are isosceles. By the Base Angle Theorem, we see that $m\angle ABO = \alpha$ and $m\angle BOC = \beta$.

Since the sum of the measures of the angles in a triangle is $180°$, we compute that $m\angle AOB = 180 - 2\alpha$, and $m\angle BOC = 180 - 2\beta$. Thus, $m\widehat{AC} = m\angle AOC = m\angle AOB + m\angle BOC = 360 - 2(\alpha + \beta)$. This shows that the measure of the dual arc, which is the included arc of the angle, is $360° - (360° + 2(\alpha+\beta) = 2(\alpha+\beta)$.

Therefore,

$$m\angle ABC = m\angle ABO + m\angle BOC = \alpha + \beta = \frac{1}{2} m\widehat{AZC}.$$

□

Proposition 21. Segment Angles
Given a circle, angles in the same segment have equal measure.

Proposition 22. Inscribed Quadrilaterals
The opposite angles of a quadrilateral inscribed in a circle are supplementary.

Proposition 32. Tangent-Chord Angles
Given a circle, a point on it, a line tangent to that point, a diameter at that point, and a chord at that point, the acute angle the chord makes with the diameter is congruent to the acute angle the chord makes with the tangent, and obtuse angle the chord makes with the diameter is congruent to the obtuse angle the chord makes with the tangent,

Proposition 33. Intersecting Chord Ratios
Given a circle and two chords which intersect inside the circle, the area of the rectangle on the two pieces of one chord has the same area as the rectangle on the two pieces of the other chord.

Proposition 35. Internal Secant-Secant Segments
Given a circle and two chords which intersect, the area of the rectangle on the segments of one chord equals the area on the rectangle on the other segments.

Proposition 36. (External Tangent-Secant Segments
Given a circle, a point outside of the circle, a line segment from the point to the circle and tangent to it, and a line segment from the point to the opposite side of the circle, the area of the square on the tangent chord equals the rectangle on the segment to the circle and the segment through the circle.